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ON A CLASS OF ORTHOGONAL SYSTEMS OF ARITHMETICAL CHARACTER

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Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be an arbitrary orthonormal system in a complex Hilbert space with inner product $\langle\ ,\ \rangle$, and $\omega(n)$ be a complex-valued, completely multiplicative function defined on the natural numbers n, i. e. a function that satisfies the condition

$$\omega(ab) = \omega(a)\,\omega(b)$$

for all natural numbers a and b.

In his paper [1] N. P. Romanov proved among other things the following two theorems:

Theorem 1. *Let the multiplicative function $\omega(n)$ satisfy the conditions*

a) $\qquad \omega(n) \neq 0 \quad$ *for all n, and*

b) $\qquad \sum_{n=1}^{\infty} |\omega(n)|^2 < \infty$.

Then the sequence

$$f_n = \frac{\overline{\omega}(n)^{-1}}{\sqrt{\sigma}} \sum_{k=1}^{\infty} \omega(k)\alpha_{kn} ,$$

where

$$\sigma = \sum_{k=1}^{\infty} |\omega(k)|^2 ,$$

possesses a D-property, or more precisely, the D_0-property

$$\langle f_m, f_n \rangle = g((m, n))$$

with the function $g(n) = |\omega(n)|^{-2}$ defined on the set of positive integers n, where (m, n) denotes as usual the greatest common divisor of m and n.

Theorem 2. *If the sequence f_n has a D-property, then the sequence*

$$\theta_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) f_d$$

is orthogonal, where $\mu(n)$ is the Möbius function.

It can be shown further that the sequence

$$\psi_n = G(n)^{-\frac{1}{2}}\theta_n$$

with $G(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d)$, is orthonormal.

In the paper [1] also proved is the theorem on completeness of the sequence ψ_n, i. e. that the ψ_n generate the same portion of the underlying Hilbert space as the α_n do. Proof of the completeness is usually carried out by means of verifying the Parseval identity, which is here always reduced to generalized identities of Euler in the theory of prime numbers. Highly numerous examples of sequences of the type $\sum_{k=1}^{\infty}\omega(k)\alpha_{kn}$ are known, where the sequence α_k is orthonormal and $\omega(n)$ is a multiplicative function nowhere equal to zero of an integral argument, with convergent sums of the squares of moduli. It will suffice to indicate the known formulas:

$$B_{2\nu}(x-[x]) = \pm 2\frac{(2\nu)!}{(2\pi)^{2\nu}}\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^{2\nu}},$$

$$B_{2\nu+1}(x-[x]) = \pm 2\frac{(2\nu+1)!}{(2\pi)^{2\nu+1}}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n^{2\nu+1}},$$

where $B_\rho(x)$ is the Bernoulli polynomial of degree ρ (cf. [2] aud [3]). However, already for the case of the well-known formulae:

$$E_{2\nu}(2x-[2x]) = \pm\frac{4\cdot(2\nu)!}{\pi^{2\nu+1}}\sum_{n=0}^{\infty}\frac{\sin(2n+1)2\pi x}{(2n+1)^{2\nu+1}},$$

$$E_{2\nu+1}(2x-[2x]) = \pm\frac{4\cdot(2\nu-1)!}{\pi^{2\nu}}\sum_{n=0}^{\infty}\frac{\cos(2n+1)2\pi x}{(2n+1)^{2\nu}},$$

the former of which can be written in the form

$$E_{2\nu}(2x-[2x]) = \pm\frac{4\cdot(2\nu)!}{\pi^{2\nu+1}}\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n^{2\nu+1}}\chi_0(n),$$

where $\chi_0(n)$ is the principal character to the modulus 4 and the coefficients $\omega(n) = \chi_0(n)n^{-\rho}$ have multiplicativity but are not everywhere different from zero, we cannot apply the method indicated above to the orthogonalization by E. Schmidt of the sequence

$$E_\rho(2nx-[2nx])\ (n=1,2,3,\cdots).\ ^{*)}$$

*) Here, the $E_\rho(x)$ are polynomials of Euler and are expressed with Bernoulli polynomials in the following form;

$$E_{\nu-1}(x) = \frac{2^\nu}{\nu}\left\{B_\nu\left(\frac{x+1}{2}\right) - B_\nu\left(\frac{x}{2}\right)\right\}$$

(cf. e. g., N. Nörlund: Vorlesungen Über Differenzenrechnung. Berlin 1924, p. 24).

The purpose of the present paper is to modify the method indicated so as to seize some cases when the condition a) is violated.

We consider the case where $\omega(n) \neq 0$ for $(n, N) = 1$ and $\omega(n) = 0$ for $(n, N) > 1$, where N is an arbitrary fixed positive integer. We examine the sequence f_n, where the indices n run through all positive integers relatively prime to N and where the f_n for $(n, N) = 1$ are defined by the formula:

$$f_n = \frac{\overline{\omega}(n)^{-1}}{\sqrt{\sigma}} \sum_{k=1}^{\infty} \omega(k)\, \alpha_{kn} = \frac{\overline{\omega}(n)^{-1}}{\sqrt{\sigma}} \sum_{(k,N)=1} \omega(k)\, \alpha_{kn}$$

with

$$\sigma = \sum_{(k,N)=1} |\omega(k)|^2 .$$

Just as in the paper [1] cited above it can be shown that

$$\langle \psi_n, \psi_m \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

where $(n, N) = (m, N) = 1$ and

$$\psi_n = G(n)^{-\frac{1}{2}} \sum_{d|n} \mu\left(\frac{n}{d}\right) f_d .$$

Here, for the construction of the orthogonal system ψ_n, use is made of the α_n only with the indices relatively prime to N, to obtain the ψ_n with the indices relatively prime to N. In order to accomplish the process of orthogonalization, in the closed hull of the sequence α_1, $\alpha_2, \cdots$ we introduce an infinite sequence of isometric linear operators T_1, $T_2, \cdots$ such that

$$T_n \left(\sum_{k=1}^{\infty} c_k \alpha_k \right) = \sum_{k=1}^{\infty} c_k \alpha_{kn} ,$$

and moreover, for every positive integer n we introduce the decomposition $n = n'n^*$, where n^* can have only such prime divisors that are contained in N and n' is relatively prime to N. The decompositions

$$a = a'a^*, \quad b = b'b^*, \quad m = m'm^* \qquad \text{etc.}$$

will have a similar meaning (the operation 'prime' and the operation 'asterisk').

Putting then $\psi_n = T_{n^*} \psi_{n'}$, we easily verify that the ψ_n, which are now defined for all n, form an orthogonal sequence in the space generated by the α_n. It is easily seen that the ψ_n can be written in the form

$$\psi_n = G(n')^{-\frac{1}{2}} \frac{1}{\sqrt{\sigma'}} \sum_{d|n} \mu\left(\frac{n}{d}\right) \chi_0\left(\frac{n}{d}\right) f_d$$

with

$$G(n') = \sum_{d|n'} \mu\left(\frac{n'}{d}\right) |\omega(d)|^{-2}, \quad \sigma' = \sum_{(k, N)=1} |\omega(k)|^{-2},$$

where χ_0 is the principal character to the modulus N. It is also easily seen that the D_g-property should be modified in the following form :

$$\langle f_n, f_m \rangle = \begin{cases} 0, & \text{if} \quad n^* \neq m^* \\ \chi_0(n' m') \, g\,((n, m)), & \text{if} \quad n^* = m^*, \end{cases}$$

where $n = n' n^*$, $m = m' m^*$ are the decompositions indicated above (cf. [4]).

Orthogonalization will be furnished by the formula

$$\theta_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \bar{\chi}\left(\frac{n}{d}\right) f_d,$$

where $\chi(n)$ is an arbitrary (not necessarily the principal) character to the modulus N.

It is easy to see that the elements appearing in our construction can be obtained from the first of them by making use of the linear operators T_n in the following way :

$$T_m \sum_{k=1}^{\infty} \omega(k)\alpha_k = \sum_{k=1}^{\infty} \omega(k)\alpha_{km};$$

$$T_m F_1 = F_m \qquad \text{with} \qquad F_m = \sum_{(k, N)=1} \omega(k)\alpha_{km}$$

since

$$f_n = \frac{\bar{\omega}(n)^{-1}}{\sqrt{\sigma}} F_n.$$

Also, in the case of the space $L^2(0, 1)$, when the $\alpha_n = \alpha_n(x)$ are given by

$$\alpha_n = \sqrt{2} \sin 2\pi n x$$

or by

$$\alpha_n = \sqrt{2} \cos 2\pi n x,$$

actions of operators T_m simply mean multiplications of the argument x

of any function subject to the action of operators, by a positive integer. Hence arises the connection of the subject of the present paper with the problem of completeness of Töplerian bases.

As is well known, a sequence

$$F(nx) \qquad (n = 1,\ 2,\ 3,\ \cdots),$$

which is complete in the sense of the metric of the underlying Hilbert space, is called a Töplerian basis of the space.

The problem of Töplerian bases is to determine the nature of the function $F(x)$ for which the sequence $F(nx)$ $(n = 1,\ 2, \cdots)$ will be complete. This problem was first formulated by a German physicist, A. Töpler, in connection with the study of electromagnetic waves. The problem has so far been resolved, with a solution far from a general one, by various authors for individual classes of functons $F(x)$.

In the present paper a new solution of this problem, not considered until now, is examined for cases of Hilbert space of functions skew-symmetric and symmetric with respect to the middle point, i. e. for the functions satisfying almost everywhere either the condition $F(1-x) = -F(x)$, or the condition $F(1-x) = F(x)$. Our schemes will give a representation of $F(x)$ solving this problem, since in the case of $\alpha_n = e^{2\pi inx}$ we have

$$T_m V(x) = V(mx),$$

where $V(x) = \sum c_n \alpha_n = \sum c_n\, e^{2\pi inx}$. It is also easy to see that from the sequence connected with the multiplicative function $\omega(n)$ in the form indicated above, one can obtain a sequence connected with the multiplicative function $\omega_0(n) = X_0(n)\,\omega(n)$, where $X_0(n)$ is the principal character to the modulus N. Indeed,

$$\sum_{d|N} \mu(d)\omega(d) \sum_{k=1}^{\infty} \omega(k)\alpha_{knd} = \sum_{k=1}^{\infty} \omega_0(k)\alpha_{kn}.$$

Thus, not only the periodicized polynomials of Bernoulli but also the functions

$$\phi_{n\,;\,\rho}(x) = n^{\rho} \sum_{d|N} \mu(d)d^{-\rho}\, B_\rho(ndx - [ndx])$$

for which a D_ρ-property, not ordinary but modified as indicated before, is valid, can serve as bases for the construction of complete systems.

We shall give an example for the case of $\rho = 1$ and $N = 4$:

$$F(x) = 2[2x] - 4[x] - 1$$

$$= -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\chi_0(k|4)}{k}\sin 2\pi kx\,;$$

$$f_n(x) = n\{2[2nx] - 4[nx] - 1\}\,;$$

$$\langle f_n,\ f_m\rangle = \begin{cases} 0, & \text{if } n \text{ and } m \text{ contain 2} \\ & \text{with different powers,} \\ (n,\ m)^2, & \text{otherwise.} \end{cases}$$

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